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ESTIMATES OF THE OPTICAL THICKNESS FOR SOME PROCESSES OF UNLIMITED COMPRESSION OF A GAS†

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The unlimited unshocked compression of an ideal gas (the unlimited increase in the density and no shock waves) is considered when the gas is at rest inside a cylinder, a sphere [1], a tetrahedron [2] and a solid of revolution with a triangular generatrix [3, 4] at the initial instant of time. It is proved that the value of the optical thickness increases without limit and the asymptotic form of its increase for instants of time close to the instant when the gas collapses to a point is obtained. Estimates are made of the ratio of the optical thickness to the energy expended in the compression.

The problem of obtaining the optical thickness [5] is related to the problem of the possibility of using intense compression processes to initiate thermonuclear synthesis (the optical thickness must reach a certain threshold value).

1. INTRODUCTION. FORMULATION OF THE PROBLEM

The properties of the well-known compression processes described previously in [1–4] are investigated, but we will henceforth only consider those aspects which are necessary for a qualitative analysis.

In the processes in question, at the initial instant of time the gas is at rest in a certain volume, which is formed by fixed walls and moving pistons. The pistons begin to move with zero velocity and, after a finite time interval, compress the gas to a point or a line. A shock wave only arises at the instant of collapse.

We will assume the gas is uniform, non-viscous and non-heat-conducting, the compression processes are adiabatic, and the equation of state is $p = p_0(\rho/\rho_0)^\gamma$. In the case of the compression of tetrahedra and bodies of conical shape we will assume $\gamma \in (1, 2)$. Without loss of generality we will assume the density and the velocity of sound of the gas at rest to be unity.

Definition of the optical thickness. We will introduce the function

$$H(S,T) = \int_{ST} \rho ds$$

where the point T is situated on the boundary of the region occupied by the gas. We will call the point S the optical centre, its law of motion is chosen so that at each instant of time the following quantity reaches a maximum

$$\min_T H(S,T) \tag{1.1}$$

and the optical thickness at the instant of time τ

$$l(\tau) = \max_{S(\tau)} \min_T \int_{ST} \rho ds$$

We will call the following quantity the optical thickness along the direction $\mathbf{n}(\|\mathbf{n}\| \neq 0)$

$$l_\mathbf{n}(\tau) = \min_{ST \| \mathbf{n}} \int_{ST} \rho ds$$

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where ST||n means that the section ST must be parallel to the vector **n**. Estimates for the optical thickness for fixed directions were obtained previously for coordinated compression of prisms in [5, 6].

In the multidimensional case (two or three spatial coordinates) the problem of finding the law of motion of the optical centre and the value of the optical thickness is extremely time consuming. It is therefore of interest to obtain at least asymptotic estimates of the optical thickness for instants of time close to the instant of collapse.

In the case of the compression of bodies of conical shape [3] the accurate solution of the equation for the velocity potential is only known for part of the compressed volume, and the equations of motion of the particles in this case can be solved in explicit form. However, the integration constants are found from the condition for the gas to be continuously next to the volume for which only a numerical solution is known. To carry out a qualitative analysis only that region is considered in which a powerful cumulative jet is formed and for which there is also an accurate solution). This simplification is justified by the fact that it is precisely in this jet that the greatest order of increase in the gas-dynamic quantities is observed. The assertion that any consideration of the remaining volume of gas does not change the estimates obtained has not been proved and is only used as a plausible hypothesis. Strictly speaking, in this case only lower limits are obtained, and the question of upper limits remains open.

A brief algorithm for obtaining estimates for multidimensional processes. Instead of obtaining the law $S(\tau)$, which makes the quantity (1.1) reach a maximum, we will specify the function $S(\tau)$. We will then obtain the lower limits for the optical thickness value on the assumption that the law of motion of the optical centre was correctly guessed. We will then prove that the order of increase in the optical thickness cannot be greater than that which is observed for the specified law $S(\tau)$.

2. ONE-DIMENSIONAL COMPRESSION PROCESSES

We can easily obtain the order of increase in the optical thickness for one-dimensional compression processes in which the gas-dynamic quantities depend on one self-similar variable. The degrees of cumulation of the gas-dynamic quantities for the compression of a cylinder and a sphere are [1]

$$r \sim (-\tau)^{\eta}, \quad p \sim (-\tau)^{-\nu\gamma\eta} \quad \rho \sim (-\tau)^{-\nu\eta}$$

$$\eta = \frac{2}{\nu(\gamma - 1) + 2} \tag{2.1}$$

($\nu = 2$ for a cylinder, $\nu = 3$ for a sphere and r is the distance to the centre of the sphere or the axis of symmetry of the cylinder). It follows from the definition of the optical thickness and formulae (2.1) that

$$l \sim \rho r \sim (-\tau)^{n_\nu}$$

$$n_2 = -\frac{1}{\gamma}, \quad n_3 = -\frac{4}{3\gamma - 1}$$

Remark. It is necessary to refine formulae (2.1). For a fixed gas particle we have

$$r \sim R(-\tau)^{\eta}, \ \rho \sim D(-\tau)^{-\nu\eta}, \qquad R, D = \text{const}$$

where $0 \leqslant R \leqslant R_{\max}$. If $r(\tau) \equiv 0$ (the particle is on the axis or at the centre of symmetry), we have $\rho \equiv \rho_0$. Hence, it can be seen that the compression of the gas is extremely non-uniform. It can be shown that the estimate of the optical thickness value does not change when the non-uniformity of the compression is taken into account.

3. SELF-SIMILAR COMPRESSION OF A TETRAHEDRON

At the initial instant of time $\tau = -1$ the gas is at rest inside a tetrahedron $ABCO$ (Fig. 1), the face ABC is the initial position of the compressing piston, and the remaining faces are fixed non-penetrable walls. The geometrical parameters of the tetrahedron the defined by the value of the adiabatic exponent (the "matched" case). We will introduce the following notation

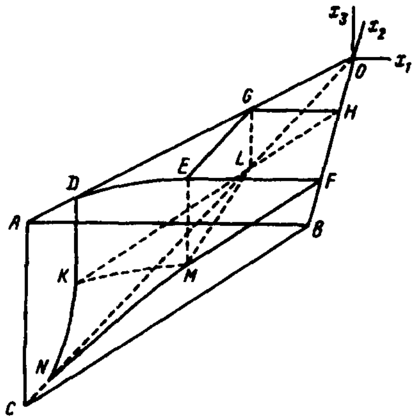

Fig. 1.

$$g = \sqrt{\frac{\gamma+1}{3-\gamma}}, \quad h = \sqrt{\frac{\gamma+1}{(2-\gamma)(3-\gamma)}}$$

The triangles ABC, AOC and BOC lie in the planes $x_2 = -1$, $x_1 = gx_2$ and $hx_1 = gx_3$ respectively. The equation of the edge CO has the form $x_1 = gx_2 = (g/h)x_3$. The configuration of the compressed volume at a certain instant of time is shown in Fig. 1. The tetrahedron $GHLO$ contains unperturbed gas. In the regions $EFMGHL$, $DGEKLM$ and $KLMN$ the gas flows are simple, double and triple self-similar waves. At the final instant of time ($\tau = 0$) the gas is compressed to a point O. the exact solution of the problem was derived earlier in [2]. (The compressed volume in Fig. 1 is confined, for clarity, along the Ox_3 axis, and the section OC may be much longer than the section AB, since the length of OC increases without limit as $\gamma \to 2$.)

The trajectories of the particles. We will introduce the notation $\mathbf{x} = (x_1, x_2, x_3)^T$, $\mathbf{u} = (u_1, u_2, u_3)^T$ for the radius vector of a gas particle and its velocity vector. In a triple wave the values of the velocity vector components define the value of the velocity of sound

$$c = 1 + \mathbf{c}_u \mathbf{u} \tag{3.1}$$

(here we have in mind the product of the row vector $\mathbf{c}_u = \sigma^1 (g, 1, h)$ and the column vector $\mathbf{u}$, $\sigma = 2(\gamma-1)^{-1}$). The gas flow is defined by the implicit formulae

$$x_i/\tau = u_i + \sigma c \partial c/\partial u_i$$

which can be written in the matrix form

$$\mathbf{x}/\tau + \mathbf{x}^0 = A\mathbf{u} \tag{3.2}$$

where $\mathbf{x}^0$ is a column vector and A is a 3×3 matrix with constant coefficients. The matrix A is similar to the diagonal matrix B, i.e. a non-degenerate matrix T exists such that $B = T^{-1}AT$. We can take the following as the transforming matrix

$$T = \begin{Vmatrix} -1/\sqrt{2-\gamma} & -g^{-1} & \sqrt{2-\gamma} \\ 0 & 1 & h^{-1} \\ 1 & 0 & 1 \end{Vmatrix}$$

The diagonal elements of matrix B are denoted by b_1, b_2 and b_3 (those are the eigenvalues of matrix A).

We will change to a new system of coordinates $Oy_1y_2y_3$, by making the linear replacement $\mathbf{y} = T^{-1}\mathbf{x}$. Relation (3.2) then takes the form

$$\mathbf{y}/\tau + \mathbf{y}^0 = B\mathbf{v} \tag{3.3}$$

$$\mathbf{v} = \frac{d\mathbf{y}}{d\tau}, \quad \mathbf{y}^0 = (y_1^0, y_2^0, y_3^0)^T = T^{-1}\mathbf{x}^0$$

The trajectories of the particles are found from the system

$$\frac{dy_i}{d\tau} = \frac{1}{b_i}\left(\frac{y_i}{\tau} + y_i^0\right), \quad i = 1,2,3 \tag{3.4}$$

$$b_1 = b_2 = 1, \quad b_3 = \frac{\gamma+1}{2(2-\gamma)}, \quad y_1^0 = y_2^0 = 0, \quad y_3^0 = h$$

The general solution of system (3.4) has the form

$$y_i = k_i\tau, \quad k_i = \frac{y_i(\tau_0)}{\tau_0}, \quad i = 1,2; \quad y_3 = \frac{y_3(\tau_0) - k\tau_0}{(-\tau_0)^b}(-\tau)^b + k\tau \tag{3.5}$$

$$k = \frac{y_3^0}{b_3 - 1}, \quad b = \frac{1}{b_3} \in (0,1), \quad \tau_0 = \text{const}$$

It is easy to verify that the Oy_3 axis coincides with the straight line CO. (The origin of coordinates remains at the point O while the point $C = (-g, -1, -h)$ in the new system has the coordinates $(0, 0, -h)$.)

When carrying out the qualitative analysis we will take into account only those parts which are in the region of the triple wave. (It will be shown that a consideration of the remaining volume of gas has no effect on the estimates obtained).

Estimate of the density. Formula (3.1) has a more compact form in the system of coordinates $Oy_1y_2y_3$. We will introduce the self-similar variables $\eta_i = y_i/\tau$, then

$$c = 1 + \mathbf{c}_u T\mathbf{v} = 1 + c_v v_3 = c_1 + c_\eta \eta_3 \tag{3.6}$$

$$c_v = \frac{1}{\sigma}\sqrt{\frac{3-\gamma}{(\gamma+1)(2-\gamma)}}, \quad c_1 = 1 + c_v bh, \quad c_\eta = c_v b$$

Hence we obtain the following estimate for the density

$$\rho = c^\sigma = D(\eta_3)\eta_3^\sigma, \quad D(\eta_3) = (c_1/\eta_3 + c_\eta)^\sigma$$
$$0 < D_l \leq D(\eta_3) \leq D_g, \quad D_l, D_g = \text{const} \tag{3.7}$$

The lower limit for the optical thickness value.

Assertion 1. At the instant τ_0 we will take two particles, whose coordinates will be denoted by (y_{11}, y_{12}, y_{13}) and $y_{21}, y_{22}, y_{23})$. If $y_{1i}(\tau_0) = y_{2i}(\tau_0)$, then when $\tau > \tau_0$ we will have the equality $y_{1i}(\tau) = y_{2i}(\tau)$.

The proof is obvious.

Consider a certain instant of time τ_0. In the region of the triple wave we will distinguish an individual volume of gas having the form of a cube $A_1 \ldots A_8$ (Fig. 2), in which the distance between the parallel faces is r_0, the faces $A_1A_2A_3A_4$ and $A_5A_6A_7A_8$ are parallel to the Oy_1y_2 the faces $A_1A_5A_8A_4$ and $A_2A_6A_7A_8$ are parallel to the Oy_1y_3 plane and the faces $A_1A_5A_6A_2$ and $A_4A_8A_7A_3$ are parallel to the Oy_2y_3 plane. By formulae (3.5) and Assertion 1, during compression this volume will take the form of a parallelepiped where the minimum distance between two opposite faces will be $\tau r_0/\tau_0$, where r_0, τ_0 = const. By formulae (3.5) the face $A_5A_6A_7A_8$ moves in accordance with the relation

$$y_3(\tau) = f(\tau) = \frac{y_3(\tau_0) - k\tau_0}{(-\tau_0)^b}(-\tau)^b + k\tau$$

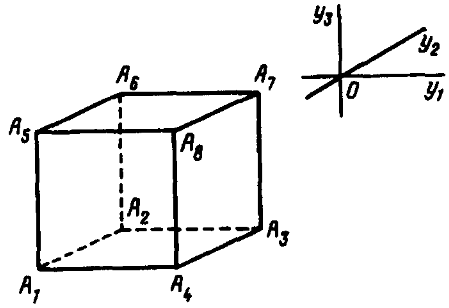

Fig. 2.

Then, for the volume considered

$$y_3(\tau) < f(\tau),\ \eta_3(\tau) > f(\tau)/\tau$$

it follows from estimate (3.7) that in this volume

$$\rho \geqslant R_0(-\tau)^{\sigma(b-1)},\ R_0 = \text{const} > 0 \tag{3.8}$$

We will take as the optical centre the centre of the parallelepiped, and the distance from the optical centre to the boundary of the volume

$$r \geqslant r_0(2\tau_0)^{-1}\tau$$

At the boundary of the parallelepiped we take the point T. The following inequality holds

$$\int_{ST} \rho ds \geqslant R_0(-\tau)^{\sigma(b-1)} \frac{r_0}{2\tau_0}\tau$$

Hence we obtain the lower limit for the optical thickness value

$$l \geqslant L_1(-\tau)^{n_l} \tag{3.9}$$

$$L_1 = \frac{R_0 r_0}{2\tau_0} = \text{const} > 0,\ n_l = \sigma(b-1)+1 = \frac{\gamma-5}{\gamma+1} < -1$$

Using the estimate of the density (3.8) it can be shown that the value of the optical thickness along the Oy_3 axis will be $O(\tau^{-2})$, $\tau \to 0$ (for the given choice of the relation $S(\tau)$).

Upper limits for the value of the optical thickness. We will prove that, in the region of a triple wave in a direction perpendicular to the Oy_3, the greatest possible order of growth of the optical thickness is $O(-\tau)^{n_l}$.

Remark. In the region *KLMN* consider the subregion

$$y_3 > (h + 2\varepsilon)\tau,\ \varepsilon = \text{const} > 0 \tag{3.10}$$

which is adjacent to the boundary of the triple wave. It follows from relation (3.6) that a constant $m > 0$ exists such that for any y_1, y_2, y_3 and τ if $y_3 > (h + 2\varepsilon)\tau$, then

$$\rho(y_1, y_2, y_3, \tau) = c^{\sigma}(y_1, y_2, y_3, \tau) < m$$

since $\eta_3 = y_3/\tau < h + 2\varepsilon$. Hence, the optical centre cannot be situated in region (3.10).

Consider a section of the compressed volume by a plane parallel to the Oy_1y_2 plane, which we will denote by P. At a fixed instant of time the gas density in this section is constant; denote its value by ρ.

Assertion 2. Suppose r_m is the greatest radius of the circle in the section P (i.e. all points inside the circle belong to P). Then

$$l = \max_{S\in P} \min_{T\in\partial P} H(S,T) = \rho r_m$$

(∂P – is the boundary of the section).

Proof. If a circle exists having the above-mentioned properties, it is obvious that $l \geqslant \rho r_m$. We will assume that we can choose a point S_0 so that

$$\min_{T\in\partial P} H(S,T) > \rho r_m$$

Consider a circle of radius r_m with centre at the point S_0. By the definition of r_m, this circle will touch the boundary of the section at a certain point T_0 and then $H(S_0, T_0) = \rho r_m$. We have obtained a contradiction.

Suppose the law of motion of the optical centre is

$$S(\tau) = (y_1(\tau), y_2(\tau), z(\tau)), \quad z(\tau) = -q(\tau)(-\tau)^b < (h+2\varepsilon)\tau$$

We will introduce the following notation: $P(\tau)$ is a section parallel to the Oy_1y_2 plane passing through the point $S(\tau)$ and $r_m(\tau)$ is the greatest radius of the circle in the section $P(\tau)$. The following relations hold

$$l_p(\tau) = \min_{T\in\partial P(\tau)} H(S(\tau),T) \leqslant -r_1(\tau)\tau D_g(q(\tau)(-\tau)^{b-1})^\sigma =$$

$$= -D_g r_1(\tau) q^\sigma(\tau)(-\tau)^{n_l}, \quad r_1(\tau) = -r_m(\tau)/\tau$$

If $\varepsilon_1 = \text{const} > 0$ exists such that $q(\tau) > \varepsilon_1$, the value of $r_1(\tau)$ is bounded.

We will prove that if $\lim_{\tau\to 0} q(\tau) = 0$, then $\lim_{\tau\to 0} r_1(\tau)q(\tau) = 0$. We will then prove that, in a direction perpendicular to the Oz axis, the greatest possible order of increase in the optical thickness is $O((-\tau)^{n_l})$.

Consider the following moving volume (which we will denote by $\Omega_0(\tau)$): a cone, the base of which is a circle of radius $r_m(\tau)$ with centre at the point $S(\tau)$, while the vertex of the cone is the point $(0, 0, z_0)$, where $z_0 = h\tau$. We will denote by $\Omega(\tau)$ the truncated cone obtained from $\Omega_0(\tau)$ by introducing a plane parallel to the Oy_1y_2 plane passing through the point $(0, 0, z_1)$, where $z_1 = (h + \varepsilon)\tau$. The mass of gas in the region $\Omega(\tau)$ is

$$m(\tau) = \iiint_{\Omega(\tau)} \rho\, dy_1 dy_2 dy_3 = \pi m_1, \quad m_1 = \int_z^{z_1} \rho\left(\frac{y_3}{\tau}\right) R^2(y_3,\tau) dy_3, \qquad R(y_3,\tau) = \frac{r_m(\tau)}{z - z_0}(y_3 - z_0)$$

We will define

$$m_2 = D_l \int_z^{z_1} \left(\frac{y_3}{\tau}\right)^\sigma \left(\frac{r_m(\tau)}{z - z_0}(y_3 - z_0)\right)^2 dy_3 \leqslant m_1$$

The value of m_2 is bounded for any τ. Further

$$m_2 = D_l \int_z^{z_1} \left(\frac{y_3}{\tau}\right)^\sigma \left(\frac{r_m(\tau)}{z - z_0}(y_3 - z_0)\right)^2 dy_3 =$$

$$= D_l(-\tau)^{2-\sigma}\left(\frac{r_1(\tau)}{z - z_0}\right)^2 \int_z^{z_1} (-y_3)^\sigma (y_3 - z_0)^2 dy_3 =$$

$$= D_l(-\tau)^{2-\sigma} r_1^2(\tau)(-z)^{-2}\left(\frac{z_0}{z} - 1\right)^{-2} \int_z^{z_1} (-y_3)^{\sigma+2}\left(\frac{z_0}{y_3} - 1\right)^2 dy_3$$

Note that the following inequalities are satisfied for $y_3 \in [z, z_1]$

$$\frac{z_0}{y_3} - 1 \leqslant \frac{z_0}{z_1} - 1 < \frac{h}{h+\varepsilon} - 1 < 0$$

Hence it follows that the quantity

$$m_3 = (-\tau)^{2-\sigma} r_1^2(\tau)(-z)^{-2} \int_z^{z_1} (-y_3)^{\sigma+2} dy_3 =$$

$$= (-\tau)^{2-\sigma} r_1^2(\tau)(-z)^{-2} \frac{1}{\sigma+3}((-z)^{\sigma+3} - (-z_1)^{\sigma+3}) =$$

$$= \frac{1}{\sigma+3}(-\tau)^{2-\sigma} r_1^2(\tau)(-z)^{\sigma+1}\left(1 - \left(\frac{z_1}{z}\right)^{\sigma+3}\right)$$

is bounded. Similarly

$$\frac{z_1}{z} < \frac{h+\varepsilon}{h+2\varepsilon} < 1$$

Consequently, the quantity

$$m_4 = (-\tau)^{2-\sigma} r_1^2(\tau)(-z)^{\sigma+1} = (-\tau)^{2-\sigma+b(\sigma+1)} r_1^2(\tau) q^{\sigma+1}(\tau) \tag{3.11}$$

is bounded. The exponent of $(-\tau)$ in formula (3.11) is equal to zero. Hence, we obtain that for any τ the quantity $r_1^2(\tau)q^{\sigma+1}(\tau)$ and, of course, the quantity $r_1(\tau)q^{(\sigma+1)/2}(\tau)$ also, are bounded.

Taking into account the relations

$$\frac{\sigma+1}{2} = \frac{1}{2}\frac{\gamma+1}{\gamma-1} < \frac{2}{\gamma-1} = \sigma$$

we conclude that

$$\lim_{\tau\to 0} r_1(\tau) q^{\sigma}(\tau) = 0$$

The effect of the double-wave region on the optical thickness value. We will first assume that the optical centre lies in the triple-wave region. We will denote by r_2 the distance from the chosen gas particle to the origin of coordinates. In the double-wave region the following asymptotic relation [2] is satisfied as $\tau\to 0$

$$r_2 \sim (-\tau)^{-n_p}, \quad n_p = \frac{3-\gamma}{\gamma+1}$$

In order that the double-wave region should have an effect on the optical thickness value in a direction perpendicular to the Oy_3 axis, it is necessary that the quantity r_2/z should be bounded (here z is the y_3 coordinate of the point $S(\tau)$). We will assume that this requirement is satisfied. We will obtain an upper limit for the optical thickness value in a direction parallel to the Oy_3 axis.

We will assume that the relation $z \sim (-\tau)^{-n_p}$ holds as $\tau \to 0$. Then

$$\eta_3 = z/\tau \sim (-\tau)^{-n_p-1}, \quad \rho \sim c_\eta^{\sigma}(-\tau)^{(-n_p-1)\sigma}$$

We obtain the following estimate for the optical thickness

$$l \sim \rho z \sim c_\eta^{\sigma}(-\tau)^{(-n_p-1)\sigma-n_p} = c_\eta^{\sigma}(-\tau)^{-1}$$

If we assume that the optical centre lies in the double-wave region, then proceeding in a similar way we obtain that the greatest order of growth of the optical thickness for directions parallel to the ABO plane is $O((-\tau)^{-1})$.

Hence, in the case of the compression of a tetrahedron we have the following estimate for the optical thickness

$$l = L(\tau)(-\tau)^{n_t}, \quad 0 < L_1 \leqslant L(\tau) \leqslant L_2 \tag{3.12}$$

$$L_1, L_2 = \text{const}, -2 < n_t < -1$$

4. THE PROCESS OF CONICAL COMPRESSION

At the initial instant of time $\tau = -1$ the gas is inside the solid of revolution, the generatrix of which is a right-angle triangle ABO (Fig. 3), where the angle ABO is $\pi/2$. The gas flow is symmetrical about the Oz axis where the "peripheral" component of the velocity is zero. We will consider only the "matched" case, when the adiabatic exponent defines the initial geometry of the compressed volume. We will denote the value of the angle OAB by α, where [4] $tg^2\alpha = (2-\gamma)/(\gamma+1)$. The line ABO is the initial position of the compressing piston, which at a certain instant of time $\tau \in (-1,0)$ takes the form $DEFHO$. At this time there is unperturbed gas in the triangle GHO. We will choose the length of the section BO to be equal to unity; hence, at the instant $\tau = 0$ the sonic perturbation arrives at the point O, and in this time all the gas is compressed to the point O. The velocity potential was obtained in [3, 4] in the class of self-similar solutions

$$\Phi(z,r,\tau) = (\tau+1)K - \tau\Psi(\xi,\eta), \quad K = \text{const} > 0$$

(the self-similar variables are $\zeta = z/\tau$, $\eta = r/\tau$). The velocity vector components $u_1 = \Phi_z = -\Psi_\zeta$, $u_2 = \Phi_r = -\Psi_\eta$ and the square of the velocity of sound is

$$c^2 = (\gamma-1)(\Psi - \xi\Psi_\xi - \eta\Psi_\eta - (\Psi_\xi^2 + \Psi_\eta^2)/2) \tag{4.1}$$

In the region DEG (EG is a characteristic, Fig. 3) we construct the exact solution

$$\Psi = \Gamma - \frac{1}{2}(\xi^2+\eta^2); \quad \Gamma = \mu^2 A(\lambda), \quad A(\lambda) = \frac{3}{4}\frac{\gamma-1}{\gamma+1}(1+\cos 2\lambda)$$

$$\xi + \xi_0 = \mu\cos\lambda, \quad \eta = \mu\sin\lambda$$

$$\xi_0 = ((\gamma-1)A(0)(1-2A(0)))^{-1/2} - (\sin\alpha)^{-1}$$

where μ and λ are polar coordinates.

We will solve the boundary-value problem numerically in the region $EFHG$ with data on the characteristics EG and GH. Henceforth we will only take into account the region DEG.

We will represent the function Ψ in the form

$$\Psi = (\zeta^2+\eta^2)A\left(\operatorname{arctg}\frac{\eta}{\zeta}\right) - \frac{1}{2}((\zeta-\xi_0)^2+\eta^2)$$

$$\zeta = \xi + \xi_0 = (z + \tau\xi_0)\tau^{-1}$$

and we will calculate its partial derivatives

$$\Psi_\xi = (2A-1)\zeta - \frac{dA}{d\lambda}\eta + \xi_0, \quad \Psi_\eta = (2A-1)\eta + \frac{dA}{d\lambda}\zeta \tag{4.2}$$

We obtain the trajectories of the particles as a function of τ, τ_0, z_0 and r_0 (the time, the initial instant of time and the initial position). From (4.2) we obtain the equations

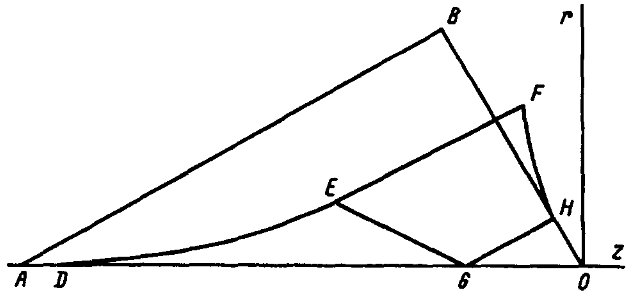

Fig. 3.

$$\frac{dz}{d\tau} = \frac{bz}{\tau} + (b-1)\xi_0, \quad \frac{dr}{d\tau} = \frac{r}{\tau} \tag{4.3}$$

solving which we obtain

$$z(\tau) = C_1(-\tau)^b - \tau\xi_0, \quad r = C_2\tau, \quad C_1, C_2 = \text{const} \tag{4.4}$$

The integration constants are found from the condition for the gas to be continuously next to the region *EFHG*, for which only a numerical solution is known.

An estimate of the density and the optical thickness. Using relations (4.1) and (4.3) we obtain a representation for the square of the velocity of sound

$$c^2 = (\gamma-1)\left(A\zeta^2 + A\eta^2 - \frac{1}{2}\zeta^2 + b\zeta^2 - \frac{1}{2}b^2\zeta^2\right) = B(\lambda)\zeta^2$$

$$B(\lambda) = A(\lambda)\,\mathrm{tg}^2\,\lambda + A(\lambda) + b - \frac{1}{2} - \frac{1}{2}b^2 = \frac{3(\gamma-1)(2-\gamma)}{(\gamma+1)^2} > 0$$

The velocity of sound and the density depend solely on the variable ζ

$$\rho(\zeta, \eta) = c^\sigma = (\zeta B)^\sigma \tag{4.5}$$

The formulae for the gas particle trajectories (4.4) and the estimate for the density (4.5) are similar to the corresponding formulae (3.5) and (3.7). Further, the procedure for obtaining asymptotic estimates repeats practically word for word the method described in the section on the compression of a tetrahedron.

Hence, in the case of the compression of bodies of conical shape the optical thickness is subject to the limit (3.12); here it is assumed that this limit does not change if we consider the part of the compressed volume for which only a numerical solution is known.

Similar limits were obtained previously for the compression of a cylinder, a sphere and also for the self-similar and non-self-similar compression of a prism [5, 6], which differ in the value of the exponent of $-\tau$ (for different types of compression, generally speaking, there are different values of n_ν). Further, instead of (3.12) we will use the abbreviated form

$$l \sim (-\tau)^{n_\nu} \tag{4.6}$$

When $1 < \gamma < 2$ the following inequalities are satisfied

$$n_t = n_c < n_3 < n_2 < n_p = \frac{\gamma - 3}{\gamma + 1} < 0$$

where n_t, n_c, n_3, n_2 and n_p are the exponents in formula (4.6) corresponding to the compression of a tetrahedron, a cone, a sphere, a cylinder and a prism.

5. COMPARISON OF THE ENERGY INPUT TO OBTAIN LARGE OPTICAL THICKNESS VALUES

In the previous section we showed that the compression of a cone and of a tetrahedron corresponds to the highest order of growth of the optical thickness value (l). However, the order of the energy input (E) for this compression is also greater than for the other types of compression considered, and hence it is interesting to obtain an estimate for the value of l/E, which will represent how economic it is to obtain large optical thickness values.

Using the estimates obtained previously [1] for the energy inputs for compressing a cylinder and a sphere $E \sim (-\tau)^{-\nu\eta(\gamma-1)}$, $\eta = 2/[\nu(\gamma-1)+2]$, a prism [2, 6] $E \sim (-\tau)^{-4(\gamma-1)/(\gamma+1)}$, and a tetrahedron and a cone [3,7] $E \sim (-\tau)^{-6(\gamma-1)/(\gamma+1)}$, we made a comparison of the exponents a_ν in the formula

$$l/E \sim (-\tau)^{a_\nu}$$

$$a_2 = \frac{2\gamma-3}{\gamma}, \quad a_3 = 2\frac{3\gamma-5}{3\gamma-1}, \quad a_c = a_t = \frac{7\gamma-11}{\gamma+1}, \quad a_p = \frac{5\gamma-7}{\gamma+1}$$

The subscript $\nu = 2, 3, c, t$, and p corresponds to the compression of a cylinder, a sphere, a cone, a tetrahedron and a prism.

The smaller the value of a_ν the more economical is the compression. The following inequalities hold

$$a_2 > a_3 \text{ for } \gamma < 3,$$
$$a_2 > a_c \text{ for } \gamma < 1 + \sqrt{2/5},$$
$$a_p > a_2$$
$$a_3 > a_c = a_t \text{ for } \gamma < 7/5,$$
$$a_p > a_c = a_t \text{ for } \gamma < 2.$$

Hence, the compression of a cone and a tetrahedron is more economic (from the point of view of obtaining large optical thickness values) only for certain ranges of values of the adiabatic exponent compared with the compression of a cylinder and a sphere.

The limited nature of the estimates based solely on asymptotic formulae should be noted, since the asymptotic form is reached quite slowly [8, 9].

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